

Define $f(\Sigma) := L_\alpha(\Sigma) + \lambda \|\Sigma\|_{1,\text{off}}$, recall that

$$\tilde{\Sigma} \in \arg \min_{\Sigma \succ 0} f(\Sigma) - \tau \log \det \Sigma.$$

We want to show $\|\tilde{\Sigma} - \Sigma^*\|_\infty \lesssim \alpha$ in this draft by consider the following optimization problem:

$$\max_{\Sigma \in S} \|\Sigma - \Sigma^*\|_\infty + \frac{4}{\alpha} \left(\tau d + f(\hat{\Sigma}^+) - f(\Sigma) \right) \quad (1)$$

where $\hat{\Sigma}^+ \in \arg \min_{\Sigma \succeq 0} f(\Sigma)$ is the positive semi-definite estimator and S is the feasible region with $S := \{\Sigma : f(\hat{\Sigma}^+) \leq f(\Sigma) \leq f(\hat{\Sigma}^+) + \tau d\}$. For simplicity, let $g(\Sigma)$ denote the objective function in (1), i.e.

$$g(\Sigma) := \|\Sigma - \Sigma^*\|_\infty + \frac{4}{\alpha} \left(\tau d + f(\hat{\Sigma}^+) - f(\Sigma) \right).$$

Lemma 1. *Let Σ be an optimal solution to (1) with $f(\Sigma) < f(\hat{\Sigma}^+) + \tau d$, then $\|\Sigma - \Sigma^*\|_\infty \lesssim \alpha$ with high probability.*

Proof: Let $(k, l) \in \arg \max_{k', l'} |\Sigma_{k'l'} - \Sigma_{k'l'}^*|$, i.e. $|\Sigma_{kl} - \Sigma_{kl}^*| = \|\Sigma - \Sigma^*\|_\infty$. For the sake of contradictory, assume that $\Delta_{kl} := \Sigma_{kl} - \Sigma_{kl}^* > 3\alpha/2$. We can see that

$$\begin{aligned} \partial g_{kl}(\Sigma) \\ = \text{Conv} \{ \pm \lambda, 1 \pm \lambda \} - \frac{4}{\alpha} \cdot \frac{1}{n} \sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il}) \end{aligned} \quad (2)$$

Since Σ lies in the interior of the feasible region, $0 \in \partial g_{kl}(\Sigma)$. Similar to our argument in Proposition 5, we have

$$(1/n) \sum_{i=1}^n 1(|\Sigma_{kl}^* - x_{ik}x_{il}| \leq \alpha/2) > 2/3$$

with high probability. Let $\mathcal{G}_{kl} := \{i \in [n] : |\Sigma_{kl}^* - x_{ik}x_{il}| \leq \alpha/2\}$.

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il}) \\ &= \frac{1}{n} \sum_{i \in \mathcal{G}_{kl}} \rho'_\alpha(\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il}) \\ & \quad + \frac{1}{n} \sum_{i \notin \mathcal{G}_{kl}} \rho'_\alpha(\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il}) \\ &> 2\alpha/3 - \alpha/3 = \alpha/3 \end{aligned}$$

where the last inequality follows because for $i \in \mathcal{G}_{kl}$, $\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il} \geq 3\alpha/2 - \alpha/2 = \alpha$. Combining this with (2), we have

$$\partial g_{kl}(\Sigma) < 1 + \lambda - \frac{4}{\alpha} \cdot \alpha/3 = \lambda - 1/3 < 0. \quad (3)$$

The last inequality follows with $\lambda \asymp \sqrt{\log d/n}$ given $n \gtrsim \log d$, which contradicts $0 \in \partial g_{kl}(\Sigma)$. Similarly, the case for $\Delta_{kl} := \Sigma_{kl} - \Sigma_{kl}^* < -3\alpha/2$ also leads to contradictory. ■

Lemma 2. *Any optimal solution Σ to (1) with $f(\Sigma) = f(\hat{\Sigma}^+) + \tau d$ satisfies $\|\Sigma - \Sigma^*\|_\infty \lesssim \alpha$ with high probability.*

Proof: Let $(k, l) \in \arg \max_{k', l'} |\Sigma_{k'l'} - \Sigma_{k'l'}^*|$, i.e. $|\Sigma_{kl} - \Sigma_{kl}^*| = \|\Sigma - \Sigma^*\|_\infty$. For the sake of contradictory, assume that $\Delta_{kl} := \Sigma_{kl} - \Sigma_{kl}^* > 3\alpha/2$. Let

$$\partial g_{kl}^-(\Sigma) := \lim_{h \rightarrow 0^+} \frac{g(\Sigma - hE_{kl}) - g(\Sigma)}{h}. \quad (4)$$

Then

$$\partial g_{kl}^-(\Sigma) = -1 \pm \lambda + \frac{4}{\alpha} \cdot \frac{1}{n} \sum_{i=1}^n \rho'_\alpha(\Delta_{kl} + \Sigma_{kl}^* - x_{ik}x_{il}) \quad (5)$$

and given Σ is an optimal solution to (1), we must have $\Sigma - hE_{kl} \in S$ for positive and sufficiently small h , thus $g(\Sigma - hE_{kl}) - g(\Sigma) \leq 0$. Hence (4) implies $\partial g_{kl}^-(\Sigma) \leq 0$

However, with the same argument as in Lemma 1, $\partial g_{kl}^-(\Sigma) > 1/3 - \lambda > 0$, which is a contradictory. ■

In the following Theorem, we combine Lemma 1 and Lemma 2 to conclude that $\|\tilde{\Sigma} - \Sigma^*\|_\infty \lesssim \alpha$.

Theorem 3. $\|\tilde{\Sigma} - \Sigma^*\|_\infty \leq 3\alpha/2 + 4\tau d/\alpha$ with high probability. By taking $0 < \tau \lesssim \alpha^2/d$, we have $\|\tilde{\Sigma} - \Sigma^*\|_\infty \lesssim \alpha$.

Proof: Recall that

$$\hat{\Sigma}^+ \in \arg \min_{\Sigma \succeq 0} f(\Sigma) \quad (6)$$

and that

$$\tilde{\Sigma} \in \arg \min_{\Sigma \succ 0} f(\Sigma) - \tau \log \det \Sigma. \quad (7)$$

We can view (7) as the log-barrier relaxation of (6), and $\tilde{\Sigma}$ as a point on the central path towards $\hat{\Sigma}^+$. Then it follows from the convergence of central path that

$$f(\tilde{\Sigma}) \leq f(\hat{\Sigma}^+) + \tau d. \quad (8)$$

Hence, by combining this with Lemma 1 and Lemma 2,

$$\begin{aligned} & \|\tilde{\Sigma} - \Sigma^*\|_\infty + \frac{4}{\alpha} \left(\tau d + f(\hat{\Sigma}^+) - f(\tilde{\Sigma}) \right) \\ & \leq 3\alpha/2 + \frac{4}{\alpha} \left(\tau d + f(\hat{\Sigma}^+) - f(\Sigma_\tau) \right) \\ & \leq 3\alpha/2 + 4\tau d/\alpha. \end{aligned}$$

where Σ_τ is any optimal solution to (1), and the last inequality follows from $f(\hat{\Sigma}^+) \leq f(\Sigma_\tau)$. Finally,

$$\|\tilde{\Sigma} - \Sigma^*\|_\infty \leq \|\tilde{\Sigma} - \Sigma^*\|_\infty + \frac{4}{\alpha} \left(\tau d + f(\hat{\Sigma}^+) - f(\tilde{\Sigma}) \right)$$

because of (8). ■

With $\|\tilde{\Sigma} - \Sigma^*\|_\infty \lesssim \alpha$, the rest of the proof follows immediately, thus $\tilde{\Sigma}$ enjoys the same statistical convergence rate as $\hat{\Sigma}^+$, which is minimax optimal.