There are a few things to mention: If you find a proof to be confusing, don't hesitate to contact me - it might be a mistake or a typo, though I have tried my best to reduce them. Also, the Lemmas and Theorems in this document sometimes omit to mention that they are only true for d sufficiently large. Another important assumption everywhere without being mentioned is that $\{v_i\}_{i=1}^{N} \cup \{a_i\}_{i=1}^{\lfloor \frac{d}{2} \rfloor}$ is linearly independent.

Recall $D(v_{k_1}, v_{k_2}, \dots, v_{k_n}) = \{q \in R^d | v_{k_1}^T q = v_{k_2}^T q = \dots = v_{k_n}^T q\}$.
Let $g(v_1, v_2, \dots, v_N)$ be the projection of v_i to $D(v_1, \dots, v_N)$. Easy to see that this doesn't vary for different i.
Let $\lambda = ||g(v_1, v_2, \dots, v_N)||$. Then we have $\lambda = \langle v_1, e \rangle = \langle v_2, e \rangle \dots = \langle v_N, e \rangle$ for some unit vector e.

Lemma 1 Assume that $|\langle v_i, v_j \rangle| \leq \beta\sqrt{\frac{log(d)}{d}}$ with some fixed $\beta > 0$ for all $1 \leq i < j \leq N$. Then $\lambda = ||g(v_1, v_2, \dots, v_N)|| \in (\frac{\sqrt{N-o(N)}}{N}, \frac{\sqrt{N+o(N)}}{N})$.

Proof: We have

$$|N\lambda| = |\langle \sum_{i=1}^{N} v_i, e \rangle| \leq ||\sum_{i=1}^{N} v_i|| = \sqrt{\sum_{i=1}^{N} \langle v_i, v_i \rangle + \sum_{i \neq j} \langle v_i, v_i \rangle}$$

$$\leq \sqrt{N + N(N-1)\beta\sqrt{\frac{log(d)}{d}}} < \sqrt{N + o(N)}$$

Therefore we have $\lambda < \frac{\sqrt{N+o(N)}}{N}$.
To show $\lambda > \frac{\sqrt{N-o(N)}}{N}$, notice that

$$g(v_1, \dots, v_N) = \arg \min_{\sum_{i=1}^{N} p_i = 1} ||\sum_{i=1}^{N} p_i v_i||$$

So

$$||g(v_1, \dots, v_N)||^2 = \min_{\sum_{i=1}^{N} p_i = 1} ||\sum_{i=1}^{N} p_i v_i||^2 \quad = \min_{\sum_{i=1}^{N} p_i = 1} \sum_{i=1}^{N} p_i^2 + \sum_{i \neq j} p_i p_j \langle v_i, v_j \rangle$$

$$\geq \min_{\sum_{i=1}^{N} p_i = 1} \sum_{i=1}^{N} p_i^2 - \sum_{i \neq j} p_i p_j \beta \sqrt{\frac{\log(d)}{d}}$$

$$= \frac{1}{N} - N(N-1)\frac{1}{N^2}\beta\sqrt{\frac{\log(d)}{d}} > \frac{N - o(N)}{N^2}$$

□

Assuming $\{v_i\}_{i=1}^N$ are linearly independent. Then there exists $x_0 \in span\{v_1, v_2, \ldots, v_N\}$ such that $v_1^T x_0 - \gamma = v_1^T x_0 - 2\gamma = \cdots = v_N^T x_0 - N\gamma = 0$. Define

$$V^T = \begin{bmatrix} v_1^T \\ v_2^T \\ \ldots \\ v_N^T \end{bmatrix}, \Gamma = \begin{bmatrix} \gamma \\ 2\gamma \\ \ldots \\ N\gamma \end{bmatrix}$$

So we have $V^T x_0 = \Gamma$, since $x_0 \in span\{v_1, v_2, \ldots, v_N\}$, we must have

$$\sigma_N ||x_0|| \leq ||\Gamma|| = \sqrt{\frac{N(N+1)(2N+1)}{6}}\gamma$$

where σ_N is the minimum singular value of V. And we know the following Theorem in the matrix book:

Theorem 2.7.1 (Lower bound). *Let $M = (\xi_{ij})_{1 \leq i \leq p; 1 \leq j \leq n}$ be an $n \times p$ Bernoulli matrix, where $1p \leq (1-\delta)n$ for some $\delta > 0$ (independent of n). Then with exponentially high probability (i.e. $1 - O(e^{-cn})$ for some $c > 0$), one has $\sigma_p(M) \geq c\sqrt{n}$, where $c > 0$ depends only on δ.*

$\sqrt{d}V$ is an $N \times d$ Bernoulli matrix. Take $\delta = \frac{1}{2}$ (other choices like $\frac{1}{3}$ are also valid) and there is a corresponding c such that $\sigma_N \geq c\sqrt{d}/\sqrt{d} = c$.
Then we have $||x_0|| \leq \sqrt{\frac{N(N+1)(2N+1)}{6}}\gamma/c = C$ with C to be a constant determined by c.

Given that

$$\gamma = \sqrt{\frac{400k\log(d)}{d}} = 20\sqrt{\frac{k\log(d)}{d}}, \qquad N = (\frac{1}{32\gamma})^{\frac{2}{3}}$$

we have that $C \approx \frac{N^{\frac{3}{2}}}{\sqrt{3}c}\gamma = \frac{1}{32\sqrt{3}c}$, that is, $||x_0|| \leq \frac{1}{32\sqrt{3}c}$.
I had some problem analyzing c because I was stuck at a step in the proof by Tau which he leaves as an exercise. But I believe that $c \geq \frac{1}{16}$. Assuming this is correct, then $C \leq \frac{1}{2\sqrt{3}}$ and $||x_0|| < 1$ is inside the ball.

The "proof" for an easier case

Write g for $g(v_1, \ldots, v_N)$. Consider the $t > 0$ such that $||x_0 - tg|| = 1$. We now aim to show that the boundary point $x_0 - tg$ is the optimal solution of $\min_{||x|| \leq 1} \hat{f}(x)$ where $\hat{f}(x) = \max_{i=1\ldots N} v_i^T x - i\gamma$.
Lemma 2 If $t > 0$ and $||x_0 - tg|| = 1$, then $\frac{1}{||g||}(1 - \frac{1}{32\sqrt{3}c}) \leq t \leq \frac{1}{||g||}(1 + \frac{1}{32\sqrt{3}c})$.
Proof: Not hard if notice that the minimum distance from x_0 to the boundary is $1 - ||x_0||$ and the maximum is $1 + ||x_0||$. □

Let $x_0 = \sum_{i=1}^{N} \lambda_i v_i$ and $g = \sum_{i=1}^{N} p_i v_i$, then $tg - x_0 = \sum_{i=1}^{N} (tp_i - \lambda_i) v_i$. Let r denote $tp - \lambda$, so $r_i = tp_i - \lambda_i$.

Lemma 3 Assuming that $|\langle v_i, v_j \rangle| \le \sqrt{\frac{10\log(d)}{d}}$ for all $i \neq j$, we have $r_i > 0$ for all i.
Proof: Assume $r_i \le 0$ for some i.
Since $\langle v_i, \sum_{j=1}^{N} p_j v_j \rangle = ||g||^2$ and $\langle v_i, \sum_{j=1}^{N} \lambda_j v_j \rangle = i\gamma$, we have

$$||g||^2 t - i\gamma = r_i + \sum_{j \neq i} \langle v_i, v_j \rangle r_j$$

Also, with $||r|| \le ||tg|| + ||x_0|| \le ||x_0|| + 1 + ||x_0|| \le \frac{1}{16\sqrt{3}c} + 1$ we have:

$$\sum_{i=1}^{N} |r_i| \le \sqrt{N}||r|| = \sqrt{N}(\frac{1}{16\sqrt{3}c} + 1)$$

Then with the results of Lemma 1 and Lemma 2:

$$\frac{1}{\sqrt{N}}(1 - \frac{1}{32\sqrt{3}c} - \frac{1}{32c}) = \frac{1}{\sqrt{N}}(1 - \frac{1}{32\sqrt{3}c} - N^{\frac{3}{2}}\gamma) \approx \frac{N - o(N)}{N\sqrt{N + o(N)}}(1 - \frac{1}{32\sqrt{3}c}) - N\gamma$$

$$\le \frac{N - o(N)}{N^2}\frac{1}{||g||}(1 - \frac{1}{32\sqrt{3}c}) - N\gamma \le \frac{N - o(N)}{N^2}t - N\gamma \le ||g||^2 t - i\gamma = r_i + \sum_{j \neq i} \langle v_i, v_j \rangle r_j$$

$$\le 0 + \sum_{j \neq i} \sqrt{\frac{10\log(d)}{d}}|r_j| \le \sqrt{\frac{10\log(d)}{d}}\sqrt{N-1}||r|| < \sqrt{\frac{10\log(d)}{d}}\sqrt{N}(\frac{1}{16\sqrt{3}c} + 1)$$

If we assume $c > \frac{1}{16}$, then we have

$$0 < \frac{1}{2\sqrt{N}}(1 - \frac{1}{2\sqrt{3}} - \frac{1}{2}) < \sqrt{\frac{10\log(d)}{d}}\sqrt{N}(\frac{1}{\sqrt{3}} + 1)$$

That is,

$$0 < \frac{1}{2}(\frac{1}{2} - \frac{1}{2\sqrt{3}}) < \sqrt{\frac{10\log(d)}{d}}N(\frac{1}{\sqrt{3}} + 1)$$

But we know $\sqrt{\frac{10\log(d)}{d}}N \to 0$ as $d \to +\infty$, contradictory. $\square$

Lemma 4: Assume that $|\langle v_i, v_j \rangle| \le \sqrt{\frac{10\log(d)}{d}}$ for all $i \neq j$. If $t > 0$ and $||x_0 - tg|| = 1$, then $x_0 - tg = \arg\min_{||x|| \le 1} \hat{f}(x)$. Consequently, $\hat{f}^* = \hat{f}(x_0 - tg) > -\frac{\sqrt{N + o(N)}}{N}(1 + \frac{1}{32\sqrt{3}c}) \sim -O(\frac{1}{\sqrt{N}})$.

Proof: Let $\partial \hat{f}(x)$ be the subgradients of $\hat{f}$ at x. Then by the first order condition of the solution to a convex problem, the boundary point $x_0 - tg$ is the solution if $tg - x_0 \in \partial \hat{f}(x_0 - tg) = \{\sum_{i=1}^{N} \alpha_i v_i | \alpha_i \geq 0, \forall i\}$. With $tg - x_0 = \sum_{i=1}^{N} r_i v_i$ and since we have shown $r_i > 0$ in Lemma 3, this is obviously true.
Let $I(v_{k_1}, v_{k_2}, \ldots, v_{k_n}) = \{x | v_{k_1}^T x - k_1\gamma = v_{k_2}^T x - k_2\gamma = \cdots = v_{k_n}^T x - k_n\gamma\}$. For the lower bound of $\hat{f}^*$, given that $x_0 - tg \in I(v_1, \ldots, v_N)$, we have that

$$\hat{f}(x_0 - tg) = v_1^T(x_0 - tg) - \gamma = -tv_1^T g = -t||g||^2$$

$$\geq -||g||(1 + \frac{1}{32\sqrt{3}c}) > -\frac{\sqrt{N + o(N)}}{N}(1 + \frac{1}{32\sqrt{3}c})$$

□

Let $F(x) = \max\{d^5||Ax||_\infty - 1, \max_{i\in[N]} v_i^T x - i\gamma\}$, then we have $F(x) \geq \hat{f}(x)$ for all x, so $F^* \geq \hat{f}^* = -O(\frac{1}{\sqrt{N}})$.

Other resulting properties

Lemma 5 Assume that $|\langle v_i, v_j\rangle| \leq \sqrt{\frac{10\log(d)}{d}}$ for all $i \neq j$. Then the minimum of $F(x)$ must be taken at the boundary of the unit ball B and is unique.

Proof: Recall that $R(v_i) = \{x \in R^d | F(x) = v_i^T x - i\gamma\}$ was defined as the "Realm" of v_i. We can generalize the definition of "Realm" to a_i's in the matrix A in the way that $R(a_i) = \{x \in R^d | F(x) = d^5 a_i^T x - 1\}$.
Assume there exists some optimal solution $x^* \in int(B)$, then with optimality condition, we have $0 \in \partial F(x^*) = Conv\{u_i | x^* \in R(u_i)\}$, where u_i can be v_i or a_i.
Given that $\{v_i\}_{i=1}^N \cup \{a_i\}_{i=1}^{\lfloor \frac{d}{2} \rfloor}$ is assumed to be linearly independent, we have that $u_i \in \partial F(x^*)$ if and only if $x^* \in R(u_i)$.
If $x^* \in R(a_i)$ and $x^* \in R(-a_i)$, then we have $d^5 a_i^T x^* - 1 = -d^5 a_i^T x^* - 1$, so $F(x^*) = d^5 a_i^T x - 1 = -1$, which contradicts $F^* \geq -O(\frac{1}{\sqrt{N}}) \to 0$.
Therefore, either $a_i \notin \partial F(x^*)$ or $-a_i \notin \partial F(x^*)$. Consequently, $\{u_i | x^* \in R(u_i)\}$ is linearly independent, which indicates that $0 \notin Conv\{u_i | x^* \in R(u_i)\}$, contradictory.
Since any optimal solution must be taken at the boundary of the unit ball, if x_1^* and x_2^* are two distinct optimal solutions, then by convexity of the problem, $\frac{x_1^* + x_2^*}{2}$ is also an optimal solution, however, $\frac{x_1^* + x_2^*}{2} \in int(B)$, contradictory. Therefore x^* is unique. □

Lemma 6: Any optimal solution x^* must be in $\bigcup_{i=1}^{N} R(v_i)$.

Proof: If there exists optimal solution $x^* \in B$ and $\epsilon > 0$ such that $B(x^*, \epsilon) \cap$

$B \cap [\bigcup_{i=1}^{N} R(v_i)] = \emptyset$, then consider $\bar{F}(x) = d^5||Ax||_\infty - 1$. Then x^* is also an optimal solution to the problem minimizing $\bar{F}$ because $\bar{F}|_{B(x^*,\epsilon)} = F|_{B(x^*,\epsilon)}$. Since we already know $\bar{F}(0) = -1 \lll -O(\frac{1}{\sqrt{N}})$, we have that $F(x^*) = \bar{F}(x^*) \leq \bar{F}(0) = -1 \lll -O(\frac{1}{\sqrt{N}})$, which contradicts the result of Lemma 4. $\square$